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Determination of the Eigenfrequencies of a Ferrite-Filled Cylindrical Cavity Resonator Using the Finite Element Method

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***Abstract*—A formulation of the Finite Element Method (FEM) particular to axisymmetric problems containing anisotropic media is compared to an analytic solution. In particular, the resonant frequencies of a longitudinally biased ferrite-filled cylindrical cavity are examined. For comparison, a solution of the characteristic equation for the lossless, ferrite-filled cylindrical waveguide was modified to give the resonant frequencies of the cylindrical cavity. This analytical solution was then used to examine the error in the FEM formulation for the anisotropic case. It is noted that the FEM formulation for anisotropic material presented, based on both node and edge-based elements, is found to be free of spurious solutions.**

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I. Introduction

It is natural to extend tangential vector finite elements to not only inhomogeneities, but to the anisotropic case. Specifically, Wang and Ida [1] were able to show that this extension could be free of spurious modes. Their method was based on the use of tetrahedral and hexahedral elements. They noted that for permeability tensors without off-diagonal terms, symmetry could be applied to simplify the analysis. This simplification, however, is not suitable for ferrite-filled cylindrical cavities. For a ferrite-filled cylindrical waveguide, Dillon *et al.* [2] applied periodic boundary conditions to solve for phase constants. This procedure reduces the order of the solution to one-half of the original three dimensional problem.

Another method of reducing computational complexity is detailed here. By applying a Fourier mode expansion to the fields in these azimuthally invariant geometries, simplification is inherent. All modal information is retained, important for the ferrite-filled cavity where the resonant frequencies of the $\pm n$ modes can differ. The FEM analysis is thus effectively reduced to two dimensions.

Axisymmetric geometries of interest in the past included circular waveguides filled with longitudinally biased ferrites. Solutions for the phase constants of these ferrite-filled circular waveguides can be modified for the ferrite-filled cavity. Application of the appropriate boundary conditions then gives a characteristic equation which is solved for the eigenfrequencies. This solution will be outlined, and used as comparison for the FEM analysis.

II. Finite Element Formulation

The tensor characterizing a longitudinally biased ferrite is given by

$$\bar{\mu} = \begin{pmatrix} \mu & -j\mu' & 0 \\ j\mu' & \mu & 0 \\ 0 & 0 & \mu_z \end{pmatrix} \tag{1}$$

where μ, μ', and μ_z are functions of frequency and the DC biasing field for a magnetized ferrite. Because of the axisymmetry of the problem, the weak form of the wave equation is written in terms of electric field components normal and transverse to the ϕ direction using first order triangular nodal elements and first order edge-based finite elements [3]. The field is then expanded as a Fourier sum over the azimuthal variable, ϕ. The basis elements for this expansion are chosen to go as $e^{jn\phi}$. This choice is necessary in order to correctly model the fields within media characterized by (1). Moreover, it also allows the eigenvalues, corresponding to the eigenfrequencies here, to be found independently for each value of n by solving the generalized eigenvalue equation for the cavity

$$[\mathbf{S}]\{a\} = k_o^2[\mathbf{T}]\{a\}. \tag{2}$$

This formalism yields sparse real-symmetric $[\mathbf{S}]$ and $[\mathbf{T}]$ matrices for lossless, Hermitian $\bar{\mu}$ tensors. Consequently, standard mathematical library routines are used to solve the generalized eigenvalue equation.

III. Analytic Characteristic Equation

The analytic characteristic equation for cylindrical ferrite waveguides is due to Kales [4]. In this section, a brief summary of its modification for the particular case of a metallic cavity is presented. The cavity under consideration has a radius of R and a length of L and is filled with a single material described by (1).

Application of the boundary conditions at the ends of the cavity requires the longitudinal electric field component, $\mathbf{E}_z$, and the transverse magnetic field component, $\vec{\mathbf{H}}_t$, to vary as $\cos(\gamma z)$ while $\mathbf{H}_z$

and $\vec{\mathbf{E}}_t$ must go as $\sin(\gamma z)$ where $\gamma = (p\pi/L)$. Clearly, there arise two distinct possibilities. First, there are the $p = 0$ modes which have neither $\mathbf{H}_z$ nor $\mathbf{E}_z$ components. Second there exists the $p > 0$ modes which have both $\mathbf{E}_z$ and $\mathbf{H}_z$ components present. These modes are named HE (EH) if they become TE (TM) modes in the limit when the off diagonal component of the ferrite tensor goes to zero.

The Maxwell equations, for the HE/EH modes, give rise to a pair of coupled wave equations in $\mathbf{E}_z(\rho, \phi)$ and $\mathbf{H}_z(\rho, \phi)$

$$\nabla_t^2 \mathbf{H}_z + c\mathbf{H}_z + d\mathbf{E}_z = 0$$
$$\nabla_t^2 \mathbf{E}_z + a\mathbf{E}_z + b\mathbf{H}_z = 0$$

where

$$a = k_o^2 \epsilon_r \mu - k_o^2 \epsilon_r \frac{\mu'^2}{\mu} - \gamma^2$$
$$b = -\gamma\omega\mu_0 \frac{\mu_z \mu'}{\mu}$$
$$c = k_o^2 \epsilon_r \mu_z - \frac{\mu_z}{\mu}\gamma^2$$
$$d = \gamma\omega\epsilon_0\epsilon_r \frac{\mu'}{\mu}.$$

This pair can be decoupled and the boundary conditions on the side walls of the cavity enforced to give the characteristic equation for the HE/EH modes

$$\left(\frac{a}{\sigma_2} - 1\right)\sqrt{\sigma_2}\frac{J_n'(\sqrt{\sigma_2}R)}{J_n(\sqrt{\sigma_2}R)} - \left(\frac{a}{\sigma_1} - 1\right)\sqrt{\sigma_1}\frac{J_n'(\sqrt{\sigma_1}R)}{J_n(\sqrt{\sigma_1}R)} + \frac{\gamma^2 n\mu'}{\mu R}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) = 0 \quad (3)$$

where σ_1 and σ_2 are given by

$$\sigma_{1,2} = \frac{(a+c) \pm \sqrt{(a-c)^2 + 4bd}}{2}.$$

Simple bisection is then used to determine the eigenfrequencies to the accuracy desired.

The $p = 0$ modes are identical to the $\mathrm{TM}_{n,m,0}$ modes of the empty cavity. Their eigenfrequencies, however, are modified due to the presence of the off-diagonal terms in the ferrite tensor

$$f_{n,m,0} = \frac{x_{n,m}}{2\pi R\sqrt{\epsilon\mu(1 - (\mu'/\mu)^2)}} \quad (4)$$

where $x_{n,m}$ is the m-th zero of the n-th order Bessel function.

IV. Numerical Results

A cavity with length to radius ratio of 2.0 and ferrite tensor permeability given by (1) will be examined. The resonant frequencies have been normalized to the lowest order empty cavity ($\bar{\mu} = \mu_0 \mathbf{I}$) resonance: the $\mathrm{TM}_{0,1,0}$ mode.

Table I shows the first 10 resonances of the cavity and the eigenfrequencies of two higher order modes as found by the FEM. We generated FEM data for two different meshes, referred to here as Mesh 1 and Mesh 2. Mesh 1 results in 360 unknowns for the $n \neq 0$ modes and 370 unknowns for the $n = 0$ modes. The difference is due to the fact that $\mathbf{E}_z$ is completely known on the axis of rotation for the $n \neq 0$ modes, where it is equal to zero. Similarly, Mesh 2 gives 3686 unknowns for the $n \neq 0$ modes and 3752 unknowns for the $n = 0$ modes. Both meshes use the following relative permeability/permittivity values: $\mu = \mu_z = 1.0$, $\mu' = 0.1$ and $\epsilon = 1.0$.

Table I shows the FEM solution agrees to within 1% of those predicted by the analytical solution for Mesh 2. Moreover, no spurious solutions are present in the solution set. We note that modes with higher values of p or m suffer more error. These modes have higher field variation in the z and ρ directions, respectively, and thus require a finer mesh in order to model this variation.

TABLE I
Comparison of Cavity Resonances Between Characteristic Equation and FEM Data from Two Different Meshes

mode	f_{res}/f_0 Char. EQ	f_{res}/f_0 FEM Mesh 1	% error	f_{res}/f_0 FEM Mesh 2	% error
$\mathrm{HE}^+_{1,1,1}$	0.9900	0.9987	0.88	0.9909	0.09
$\mathrm{TM}_{0,1,0}$	1.0050	1.0040	0.10	1.0050	0.00
$\mathrm{HE}^-_{1,1,1}$	1.0257	1.0342	0.83	1.0266	0.09
$\mathrm{EH}_{0,1,1}$	1.1991	1.2092	0.84	1.2002	0.09
$\mathrm{HE}^+_{2,1,1}$	1.4177	1.4261	0.59	1.4187	0.07
$\mathrm{HE}^-_{2,1,1}$	1.4403	1.4492	0.62	1.4414	0.08
$\mathrm{HE}^+_{1,1,2}$	1.4697	1.4943	1.67	1.4719	0.15
$\mathrm{HE}^-_{1,1,2}$	1.5643	1.5853	1.34	1.5664	0.13
$\mathrm{TM}^+_{1,1,0}$	1.6014	1.6119	0.66	1.6033	0.12
$\mathrm{TM}^-_{1,1,0}$	1.6014	1.6134	0.75	1.6033	0.12
	⋮	⋮		⋮	
$\mathrm{EH}_{0,1,4}$	2.7682	2.8303	2.24	2.7733	0.18
$\mathrm{HE}_{0,3,1}$	2.9903	3.0666	2.55	3.0182	0.93
	⋮	⋮		⋮	

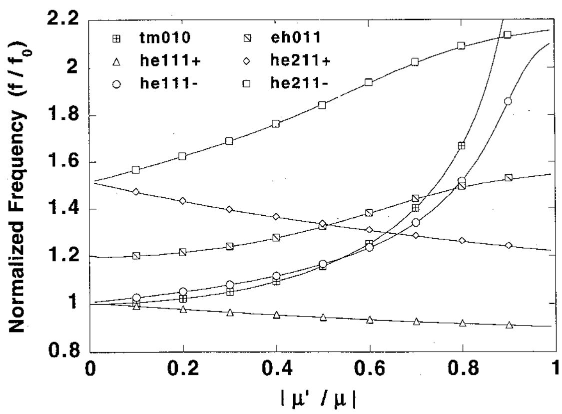

Fig. 1. Normalized resonant frequencies of the first six modes of the ferrite-filled cylindrical cavity as a function of the ratio of $|\mu'/\mu|$.

Fig. 1 exhibits the resonant frequencies of the first six modes as a function of the ratio $|\mu'/\mu|$. This ratio is, of course, a function of the applied dc biasing field and material parameters of the ferrite. The solid line curves come from the characteristic equation and are shown for comparison to the symbols which represent the FEM solutions. The relative permeability/permittivity values used are: $\epsilon = 1.0$, $\mu = 1.0$, and $\mu_z = 1.0$. Because of the frequency normalization, this figure describes all filled cavities with a length to radius ratio, $L/R = 2.0$.

In practice, it is necessary to determine the resonances of the $L/R = 2.0$ cavity when it is filled with a frequency dependent ferrite material. The relevant ferrite equations describing this dependence are

$$|\mu'/\mu| = \frac{\omega\omega_M}{\omega_0^2 - \omega^2 - \omega_0\omega_M} \quad (5)$$

and

$$\omega_M = \gamma 4\pi M_0 \qquad \omega_0 = \gamma H_0. \quad (6)$$

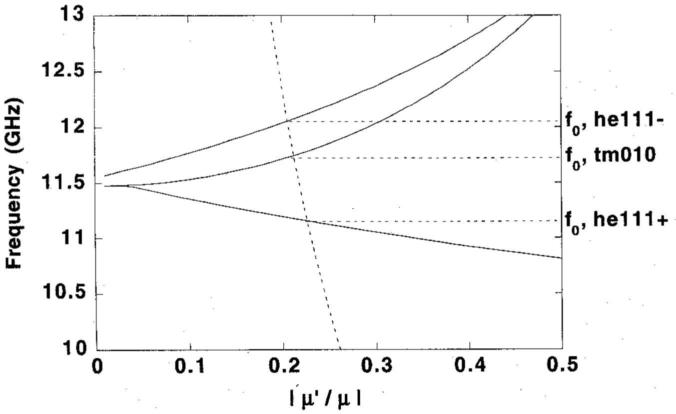

Fig. 2. Resonant frequencies of a $L = 2.0$ cm, $R = 1.0$ cm ferrite-filled cylindrical cavity.

Where H_0 is the bias field, M_0 is the magnetization and γ is the gyrotropic ratio for this material. In Fig. 2, the dashed line corresponds to a commercially available ferrite described by $\gamma = 17.6$ Mrad/(sec-Oersted), $4\pi M_0 = 800$ Gauss, and $H_0 = 1000$ Oersted. The intersections of the dashed line with the various solid lines gives the resonant frequencies of each mode. The radius of the cavity is 1.0 cm, which corresponds to a normalization frequency of $f_0 = 11.475$ GHz. Note the slope of the dashed line indicates operation above the ferromagnetic resonance.

V. Conclusion

We have determined the eigenfrequencies of a ferrite-filled cylindrical resonator using a Finite Element Method (FEM) formulation which exploits the inherent axisymmetry of the problem. This method expands the electric field using both node and edge-based elements on a two-dimensional mesh. The resulting solutions are compared to the analytical eigenvalues and are found to be free of spurious modes.

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